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ON TRIGONOMETRIC SERIES.*

By William LeRoy Hart.

1. Introduction. The present paper is concerned with series of the form

$$(1)\quad f(t) = \sum_{k=0}^{\infty} a_k \cos 2\pi\alpha_k t + b_k \sin 2\pi\alpha_k t \qquad (\alpha_0 = 0,\ \alpha_k > 0;\ k = 1, 2, \cdots),$$

where the α_k $(k = 1, 2, \cdots)$ are not necessarily commensurable, which is a generalization of Fourier series. In problems in applied mathematics where such expressions arise, the $f(t)$ are given by observation of natural phenomena or by experiment; the α_k are in some cases given by theory and, in others, remain to be determined. Thus there are two cases which it is proper to consider; first, that of the determination of the (a_k, b_k) when $f(t)$ and the α_k are given and, second, that of the determination of the (α_k, a_k, b_k) when only the $f(t)$ is known.

An example of the first case is found in the recent determination of the rigidity of the earth by A. A. Michelson† from a consideration of the tidal effects of the sun and moon on the earth. The height of the tide is known to be expansible in the form (1) and, moreover, the mathematical theory of the tides gives the values of the α_k. The determination of the (a_k, b_k) for the principal terms was the main object of the experiment which Michelson carried out.

The variation of sun-spot activity furnishes an example of the second case. The number and dimensions of sun spots vary greatly with the time, and it has been suggested by many astronomers that they may be the result of component periodic causes. In other words, it is suspected that sun-spot activity as a function of the time can be represented in the form (1). The problem then is to find the (α_k, a_k, b_k) from the curve of observation.

In article 2, it is assumed that $f(t)$ is expansible in the form (1) with known α_k and, under certain hypotheses as to the convergence of the series, it is shown how to compute the (a_k, b_k). In article 3 it is then shown that if there exists one expansion for $f(t)$ satisfying the conditions of article 2, this expansion is unique.

Professor A. Schuster has introduced the method of periodogram analysis for the determination of the periods of the unknown factors which enter into a certain compound periodic effect. In article 4, the results of the present paper are connected with a certain phase of Schuster's work.

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* Presented to the American Mathematical Society at Chicago, April 12, 1914.

† Astrophysical Journal, vol. 39 (1914), p. 105.

2. Determination of the Coefficients. The results of this article are stated in the following

Theorem I. *Let there be given the series*

$$F(t) = \sum_{n=0}^{\infty} A_n\phi_n(2\pi\alpha_n t) \qquad (\phi_0 = 1,\ \alpha_n > 0), \tag{2}$$

where $\phi_n(\theta)$ *represents either sin* θ *or cos* θ. *Then, if*

$$\sum_{n=0}^{\infty}|A_n| \tag{3}$$

converges, it follows that

$$A_i = 2\alpha_i \lim_{k=\infty}\frac{1}{k}\int_0^{k/\alpha_i} F(t)\phi_i(2\pi\alpha_i t)\,dt \qquad (i = 1, 2, \cdots),$$

$$A_0 = \delta \lim_{k=\infty}\frac{1}{k}\int_0^{k/\delta} F(t)\,dt \qquad (\delta > 0).$$

In the proof of this theorem there is needed the

Lemma 1. *If* $\alpha > 0$, $\beta > 0$ *and* $\beta \neq m\alpha$ (*m an integer*), *then*

$$\lim_{k=\infty}\frac{1}{k+1}\sum_{j=0}^{k}\phi\left[2\pi\beta\left(t+\frac{j}{\alpha}\right)\right] = 0,$$

uniformly for all t.

Let the proof of the lemma be considered only for the case when $\phi(\theta)$ is *cos* θ. The reasoning is similar for the case when $\phi(\theta) = \sin\theta$. Let

$$\begin{aligned} P_k(t) &= \sum_{j=0}^{k}\phi\left[2\pi\beta\left(t+\frac{j}{\alpha}\right)\right] = \sum_{j=0}^{k}\cos 2\pi\beta\left(t+\frac{j}{\alpha}\right), \\ &= \cos 2\pi\beta t\sum_{j=0}^{k}\cos\frac{2j\pi\beta}{\alpha} - \sin 2\pi\beta t\sum_{j=1}^{k}\sin\frac{2j\pi\beta}{\alpha}. \end{aligned} \tag{4}$$

By means of the trigonometric identities

$$\frac{1}{2}+\sum_{j=1}^{k}\cos jx = \frac{\cos kx - \cos(k+1)x}{2(1-\cos x)} \qquad (x \neq 2m\pi),$$

$$\sum_{j=1}^{k}\sin jx = \frac{\sin kx - \sin(k+1)x + \sin x}{2(1-\cos x)} \qquad (x \neq 2m\pi),$$

equation (4) becomes

$$\begin{aligned} P_k(t) &= \frac{\cos 2\pi\beta t}{2\left(1-\cos\frac{2\pi\beta}{\alpha}\right)}\left[1-\cos\frac{2\pi\beta}{\alpha}+\cos k\frac{2\pi\beta}{\alpha}-\cos(k+1)\frac{2\pi\beta}{\alpha}\right] \\ &\quad - \frac{\sin 2\pi\beta t}{2\left(1-\cos\frac{2\pi\beta}{\alpha}\right)}\left[\sin k\frac{2\pi\beta}{\alpha}-\sin(k+1)\frac{2\pi\beta}{\alpha}+\sin\frac{2\pi\beta}{\alpha}\right]. \end{aligned} \tag{5}$$

From (5) it is seen that

$$|P_k(t)| \leqq \frac{7}{2\left|1-\cos\dfrac{2\pi\beta}{\alpha}\right|} = M \qquad (k = 1, 2, \cdots; 0 \leqq t < \infty),$$

and therefore the lemma follows immediately.

To establish the theorem, it will first be proved that *if* $K_i(t)$ *represents the sum of all terms of* $F(t)$ *with* $\alpha_h = m_h\alpha_i$ *(*m_h *an integer), then*

$$\lim_{k=\infty}\frac{1}{k}\sum_{j=0}^{k-1}F\left(t+\frac{j}{\alpha_i}\right) = K_i(t), \tag{6}$$

uniformly for all t.

From the definition of $K_i(t)$ it follows that

$$K_i(t) = \sum_{h=0}^{\infty}C_h\psi_h(2\pi m_h\alpha_i t)$$

($\psi_h(\theta)$ *is either* $\sin\theta$ *or* $\cos\theta$), where the C_h are a subset of the A_n; therefore since $\sum_{h=0}^{\infty}|C_h|$ exists, $K_i(t)$ is uniformly convergent for all t. Moreover, K_i has the period $1/\alpha_i$. Hence it is seen that

$$\frac{1}{k}\sum_{j=0}^{k-1}K_i\left(t+\frac{j}{\alpha_i}\right) = K_i(t) \qquad (i = 1, 2, \cdots). \tag{7}$$

In view of (7) and the absolute convergence of all series involved, it follows that

$$\begin{aligned}\left|\frac{1}{k}\sum_{j=0}^{k-1}F\left(t+\frac{j}{\alpha_i}\right) - K_i(t)\right| &= \left|\frac{1}{k}\sum_{j=0}^{k-1}\left[F\left(t+\frac{j}{\alpha_i}\right)-K_i\left(t+\frac{j}{\alpha_i}\right)\right]\right|,\\ &= \left|\frac{1}{k}\sum_{j=0}^{k-1}H_i\left(t+\frac{j}{\alpha_i}\right)\right|,\end{aligned} \tag{8}$$

where $H_i(t)$ is the sum of all terms of $F(t)$ not contained in $K_i(t)$.

To establish (6) it must be shown that, for every value $l > 0$, expression (8) is at most equal to l if $k \geqq k_l$ (k_l chosen sufficiently large). It is easily verified that

$$H_i(t) = \sum_{m=0}^{\infty}D_m\Phi_m(2\pi\gamma_m t) \qquad (\gamma_m \neq p_m\alpha_i;\ p_m \text{ an integer}),$$

where the (D_m) are a subset of the (A_n) and where $\Phi_m(\theta)$ represents either $\sin\theta$ or $\cos\theta$. Therefore $H_i(t)$ converges uniformly for all t. Hence for every $l > 0$, h can be chosen so large that

$$\left|\sum_{m=h+1}^{\infty}D_m\Phi_m(2\pi\gamma_m t)\right| \leqq \frac{l}{2} \qquad (0 \leqq t < \infty). \tag{9}$$

By use of (9) and by an application of Lemma (1) to each term of $\sum_{m=0}^{h} D_m \Phi_m(2\pi\gamma_m t)$, it follows that k_l can be chosen so large that expression (8) is at most l for all values of t. Hence (6) is established.

From the definition of $K_i(t)$ it follows that $A_i\phi_i(2\pi\alpha_i t)$ is one of its terms. Therefore, by the theory of Fourier series, it is seen that

$$A_i = 2\alpha_i \int_0^{1/\alpha_i} K_i(t)\phi_i(2\pi\alpha_i t)\,dt.$$

Since the limit in (6) exists uniformly, it follows that

$$\begin{aligned} A_i &= 2\alpha_i \int_0^{1/\alpha_i} \left\{ \lim_{k=\infty} \frac{1}{k} \sum_{j=0}^{k-1} F\left(t + \frac{j}{\alpha_i}\right) \phi_i \left[2\pi\alpha_i \left(t + \frac{j}{\alpha_i}\right)\right]\right\} dt, \\ &= 2\alpha_i \lim_{k=\infty} \frac{1}{k} \int_0^{1/\alpha_i} \sum_{j=0}^{k-1} F\left(t + \frac{j}{\alpha_i}\right) \phi_i \left[2\pi\alpha_i \left(t + \frac{j}{\alpha_i}\right)\right] dt, \\ &= 2\alpha_i \lim_{k=\infty} \frac{1}{k} \int_0^{k/\alpha_i} F(t)\phi_i(2\pi\alpha_i t)\,dt, \end{aligned}$$

which is the result stated in the theorem.

Now consider the constant term in (2). For every given $\delta > 0$, the expression $K_\delta(t)$, the sum of the terms of $F(t)$ with α_n a multiple of δ, has A_0 as its constant term. Hence it follows that, for every $\delta > 0$,

$$A_0 = \delta \int_0^{1/\delta} K_\delta(t)\,dt = \delta \lim_{k=\infty} \frac{1}{k} \int_0^{k/\delta} F(t)\,dt.$$

It is natural to inquire what sort of functions can be expanded in the form (2). Results of a negative character are obtained in the present paper in regard to this question.

Corollary (1). *If $F(t)$ is such that $|F(t)|$ is integrable to infinity, then, unless F is identically zero, it cannot be expanded in the form* (2).

The corollary results from the fact that, for such a function F, all the coefficients in the expansion would be zero, as can be seen from the values as given in the theorem. By similar reasoning, it is seen that

Corollary (2). *If $F(t)\phi_i(2\pi\alpha_i t)$ is integrable to infinity then $A_i = 0$.*

3. Uniqueness of the expansion. In all applications of (2) it is important to know whether two series with different (α_n, A_n) can represent the same function. This question is answered in the negative in

Theorem II. *Suppose that for a given $f(t)$ there are the two expansions satisfying the hypotheses of Theorem I,*

$$f(t) = \sum_{m=0}^{\infty} A_m \phi_m(2\pi\alpha_m t) \qquad (\phi_0 = 1), \tag{10}$$

$$f(t) = \sum_{m=0}^{\infty} B_m \psi_m(2\pi\beta_m t) \qquad (\psi_0 = 1), \tag{11}$$

where $\phi_m(\theta)$ and $\psi_m(\theta)$ represent either sin θ or cos θ. Then it follows that the set (α_m) are merely a permutation of the (β_m) and that the coefficients of corresponding terms are equal. That is, the expansion of a function in the form (2) *is unique.*

By a computation similar to that used to establish (6), it follows that, because of (10),

$$\lim_{k=\infty} \frac{1}{k} \sum_{j=0}^{k-1} f\left(t + \frac{j}{\theta}\right) = A_0 + F(t, \theta) \qquad (\theta > 0), \tag{12}$$

where $F(t, \theta)$ consists of all terms of the series (10) with α_m an integral multiple of θ. Similarly it is seen that this same expression is equal to

$$B_0 + H(t, \theta), \tag{13}$$

where $H(t, \theta)$ consists of all terms of (11) with β_m an integral multiple of θ. From (12) and (13) for $\theta = \alpha_i$, and $\theta = \beta_j$, there results

$$A_0 = B_0, \qquad H(t, \alpha_i) = F(t, \alpha_i), \qquad H(t, \beta_j) = F(t, \beta_j).$$

Therefore it follows that if there is a term $A_i\phi_i(2\pi\alpha_i t)$ in (10), it is also present in (11); and if $B_j\psi_j(2\pi\beta_j t)$ is a term of (11), it is also a term of (10). Thus the theorem is established.

4. On periodogram analysis. In connection with a study of sun-spot variation, rainfall variation, etc., Schuster* was led to develop the principal of periodogram analysis for the determination of the periodic effects present in the complex resultants which are given by observation of natural phenomena. That is, on assuming that a certain effect can be represented for all values of t in a series of the form (1), Schuster's method has as its purpose the determination of the values of the unknown periods from a consideration of the curve of observation. The method given below is essentially equivalent to that of Schuster. The present treatment contains a set of hypotheses which can be taken as a basis for periodogram analysis and the results are obtained by simple deductions from the results of the preceeding articles.

Let there be given a function $f(t)$ $(0 \leqq t < \infty)$ and suppose that it can be represented in the form (1) in which it is sought to determine the (α_k, a_k, b_k). Let the function $S(\theta)$ be defined by the equations

$$S(\theta) = \lim_{m=\infty} S_m(\theta),$$

$$S_m(\theta) = \frac{2}{m\theta}\left\{\left[\int_0^{m\theta} f(t)\cos\frac{2\pi t}{\theta}\,dt\right]^2 + \left[\int_0^{m\theta} f(t)\sin\frac{2\pi t}{\theta}\,dt\right]^2\right\}^{\frac{1}{2}}.$$

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* Proceedings of the Cambridge Philosophical Society, vol. 18 (1900), p. 107.

Suppose that $\sum_{k=0}^{\infty} |a_k| + |b_k|$ converges. Then it follows that

$$S\left(\frac{1}{\alpha_i}\right) = \sqrt{a_i^2 + b_i^2} \qquad (i = 1, 2, \cdots).$$

Moreover, if $1/\theta \neq \alpha_i$ $(i = 1, 2, \cdots)$, then $S(\theta) = 0$. Hence, the graph of $S(\theta)$, which depends only on the given function $f(t)$, determines the location of the α_i.

In practice it is, of course, impossible to form the function $S(\theta)$, just as it is impossible in general to write down the sum of an infinite series. For example, in dealing with sun-spot variation, there are at hand only the data for the last 160 years. Hence, in practice, all that can be done is to compute the graph of $S_m(\theta)$ where m can be taken large for small values of θ but must be taken small for large values of θ, and is undefined for all values of m if θ is sufficiently large. However, if $S_m(\theta)$ should be graphed for a certain range $(\theta_0 \leqq \theta \leqq \theta_1)$, its form for m sufficiently large would give an indication of the value of $S(\theta)$ for $(\theta_0 \leqq \theta \leqq \theta_1)$. Hence, a maximum of $S_m(\theta)$ at $\theta = a$ shows that $S(a) \neq 0$ and therefore indicates the presence of an α_k in (1) with the value $\alpha_k = 1/a$. The graph of $S_m(\theta)$ for a given function is identical with the periodogram defined by Schuster, and the determination of the periods from the maxima is his method.

University of Chicago,
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